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Foias Numbers

Nicolae Anghel

Abstract

The Foias constant, a true mathematical gem, is generalized to a host of similar numbers. As is the case with all significant mathematics it is the underlying method, due to Foias, that matters.

The Foias constant is the unique positive real number x_1 for which the recursion

$$x_{n+1} = \left(1 + \frac{1}{x_n}\right)^n, \quad x_1 > 0, \quad n = 1, 2, \dots \tag{1}$$

has the property that $\lim_{n\to\infty} x_n = \infty$. Very likely transcendental, its 15-digit approximation is 1.187452351126501 [4, 7]. For the fascinatingly bizarre story of the discovery of this number and a proof, please see [2].

In this note we identify the principal features in Foias' original proof and streamline them into a general result. Our result gives further evidence to the observation [2] that the connection between the sequence (1) with initial seed the Foias constant and the Prime Number Theorem [1] must be fortuitous.

Theorem. *Let* $(f_n)_{n=1}^{\infty}$ *be a sequence of strictly decreasing continuously differentiable functions with increasing non-vanishing derivatives, all with the same domain,* $(0,\infty)$*, and the same range,* (r,∞)*,* $r \geq 0$*, and such that*

$$\Sigma := \bigcap_{n=1}^{\infty} \left(f_n \circ f_{n-1} \circ \cdots \circ f_1\right)(0,\infty) \neq \emptyset. \tag{2}$$

Assume that there is a number $c \in \Sigma$ *such that*

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(i) $\lim_{n\to\infty} f_n(c) = \infty$.

(ii) $\lim_{n\to\infty} \alpha_n = \infty$, *where* $\alpha_n := \left(f_{n+1}^{-1} \circ f_{n+2}^{-1}\right)(c)$, $n = 0, 1, \ldots$

(iii) $\lim_{n\to\infty} \left(f_n^{-1}\right)'(\alpha_n) = 0$.

(iv) $\liminf_{n\to\infty} \dfrac{2^n}{f_n^{-1}(c)} > 0$.

Then the recursion

$$x_{n+1} = f_n(x_n), \quad x_1 > 0, \quad n = 1, 2, \ldots, \tag{3}$$

has the Foias property, namely there is an unique $x_1 > 0$, *called the Foias number associated to the sequence of functions* $(f_n)_n$ *and denoted by* $\mathbf{c}$, *such that* $\lim_{n\to\infty} x_n = \infty$. *Moreover,*

$$\mathbf{c} = \lim_{n\to\infty} \left(f_1^{-1} \circ f_2^{-1} \circ \cdots \circ f_n^{-1}\right)(c). \tag{4}$$

Examples of sequences $(f_n)_n$ satisfying the hypotheses of the theorem are

$$\begin{aligned} &f_n(x) = \left(1 + \frac{1}{x}\right)^n, \ r = 1, \ \Sigma = \left(\frac{\left(\left(5^3 + 4^3\right)^4 + 5^{12}\right)^5}{\left(5^3 + 4^3\right)^{20}}, 4 \right), \\ &c = e, \quad \alpha_n = \frac{1}{\sqrt[n+1]{1/\left(\sqrt[n+2]{e} - 1\right) - 1}}, \end{aligned} \tag{5}$$

or

$$f_n(x) = e^{n/x}, \ r = 1, \ \Sigma = \left(e^{3e^{-2}}, e^2\right), \ c = e, \quad \alpha_n = \frac{n+1}{\ln(n+2)}. \tag{6}$$

Notice that (5) is Foias' case. An example of a sequence which might, however does not, satisfy the hypotheses or the conclusion of the theorem is

$$f_n(x) = \frac{n}{x}, \ r = 0, \ \Sigma = (0, \infty), \quad \text{no } c \text{ exists}. \tag{7}$$

In this case, $(x_n)_n$ always diverges to ∞.

In order to prove the Theorem we need some preparation. Fixing $t > 0$ and a positive integer n, define a sequence $(t_{n+m+1})_{m=1}^{\infty}$ depending on t by

$$t_{n+m+1} := (f_{n+m} \circ \cdots \circ f_n)(t). \tag{8}$$

We say that the sequence $(t_{n+m+1})_m$ has property (A) if its odd-term subsequence $(t_{n+2m})_m$ is bounded above while its even-term subsequence $(t_{n+2m+1})_m$ diverges to ∞. Similarly, it has property (B) if the words odd-even are switched in property (A).

Lemma. *In the above setting, and assuming the hypotheses* (i) *and* (ii) *of the Theorem, for any* $n = 1, 2, \ldots,$ *define the disjoint subsets* A_n *and* B_n *of* $(0, \infty)$ *by*

$$\begin{aligned} A_n &:= \{\, t \mid (t_{n+m+1})_m \quad \text{has property (A)} \,\} \\ B_n &:= \{\, t \mid (t_{n+m+1})_m \quad \text{has property (B)} \,\} \end{aligned} \tag{9}$$

Then the following is true about the sets A_n *and* B_n*:*

(v) $f_n(A_n) = B_{n+1} \cap (r, \infty)$ *and* $f_n(B_n) = A_{n+1} \cap (r, \infty)$.

(vi) If $t \in A_n$ *and* $t' \in B_n$ *then* $(0, t] \subseteq A_n$, $[t', \infty) \subseteq B_n$, *and* $t < t'$.

(vii) If n *is such that* $\alpha_k \geq c$ *for* $k \geq n+1$ *then* $\alpha_{n-1} \in A_n$ *and* $f_n^{-1}(\alpha_n) \in B_n$. *Consequently,* $\alpha_{n-1} < f_n^{-1}(\alpha_n)$.

Proof. (v) follows immediately from (8) and the very definitions (9) of the sets A_n and B_n, while the monotonicity properties of the functions f_k's, namely the composition of an even/odd number of f_k's is strictly increasing/decreasing, implies (vi).

To the end of proving (vii) notice first that since $c \in \Sigma$, α_n is well-defined. In particular, $\alpha_n > r, n = 1, 2, \ldots.$ The premise of (vii) always holds true for n sufficiently large, by (ii). Set now $t = \alpha_{n-1}$ for an allowable n. Then,

$$t_{n+2} = (f_{n+1} \circ f_n)(\alpha_{n-1}) = c,$$

since the definition of α_{n-1} in (ii) is equivalent to $(f_{n+1} \circ f_n)(\alpha_{n-1}) = c$. Thus, $t_{n+2} = c \leq \alpha_{n+1}$ yields $t_{n+2} \leq \alpha_{n+1}$, and repeating the above procedure gives

$$t_{n+4} = (f_{n+3} \circ f_{n+2})(t_{n+2}) \leq (f_{n+3} \circ f_{n+2})(\alpha_{n+1}) = c \leq \alpha_{n+3}.$$

More generally, by iteration

$$t_{n+2m} \leq c \leq \alpha_{n+2m-1}, \quad m = 1, 2, \ldots \tag{10}$$

Now, (10) implies

$$t_{n+2m+1} = f_{n+2m}(t_{n+2m}) \geq f_{n+2m}(c). \tag{11}$$

From (10), (11), and (i) it follows that the sequence $(t_{n+m+1})_m$ has property (A) for $t = \alpha_{n-1}$, and so $\alpha_{n-1} \in A_n$.

Finally, we have $\alpha_n \in A_{n+1}$ too, which is equivalent to $f_n^{-1}(\alpha_n) \in B_n$, by (vi), and $\alpha_{n-1} < f_n^{-1}(\alpha_n)$ follows also from (vi). $\square$

Proof of the Theorem. In view of the Lemma there are real numbers a_n, b_n, $n = 1, 2, \dots$, such that

$$(0, a_n) \subseteq A_n \subseteq (0, a_n] \quad \text{and} \quad (b_n, \infty) \subseteq B_n \subseteq [b_n, \infty). \tag{12}$$

Also, for n large enough, more precisely for n such that $\alpha_k \geq c$, $k = n+1, n+2, \dots$, we have

$$\alpha_{n-1} \leq a_n \leq b_n \leq f_n^{-1}(\alpha_n), \quad \text{and} \quad f_n(a_n) = b_{n+1},\ f_n(b_n) = a_{n+1}, \tag{13}$$

from which it follows that

$$\begin{aligned} \frac{b_n - a_n}{f_{n+1}^{-1}(c)} \leq & \frac{f_n^{-1}(\alpha_n) - \alpha_{n-1}}{f_{n+1}^{-1}(c)} = \frac{f_n^{-1}(\alpha_n) - f_n^{-1}\left(f_n(\alpha_{n-1})\right)}{f_{n+1}^{-1}(c)} = \\ & \frac{f_n^{-1}(\alpha_n) - f_n^{-1}\left(f_{n+1}^{-1}(c)\right)}{f_{n+1}^{-1}(c)}. \end{aligned} \tag{14}$$

By (i) we can also assume that

$$f_{n+2}(c) \geq c, \quad \text{or} \quad c \leq f_{n+2}^{-1}(c), \quad \text{or} \quad f_{n+1}^{-1}(c) \geq f_{n+1}^{-1} \circ f_{n+2}^{-1}(c) = \alpha_n.$$

f_n' and $\left(f_n^{-1}\right)'$ are increasing functions taking only negative values, and so $|f_n'| = -f_n'$ and $\left|\left(f_n^{-1}\right)'\right| = -\left(f_n^{-1}\right)'$ are decreasing. An application of the mean value theorem on the interval $\left[\alpha_n, f_{n+1}^{-1}(c)\right]$ gives

$$\begin{aligned} f_n^{-1}(\alpha_n) - f_n^{-1}\left(f_{n+1}^{-1}(c)\right) \leq & \left|\left(f_n^{-1}\right)'(\alpha_n)\right| \left(f_{n+1}^{-1}(c) - \alpha_n\right) < \\ & \left|\left(f_n^{-1}\right)'(\alpha_n)\right| f_{n+1}^{-1}(c). \end{aligned} \tag{15}$$

Putting together (14) and (15) leads to

$$\frac{b_n - a_n}{f_{n+1}^{-1}(c)} < \left|\left(f_n^{-1}\right)'(\alpha_n)\right|. \tag{16}$$

Also, since $b_n \leq f_n^{-1}(\alpha_n)$,

$$|f_n'(b_n)| \geq \left|f_n'\left(f_n^{-1}(\alpha_n)\right)\right| = \frac{1}{\left|\left(f_n^{-1}\right)'(\alpha_n)\right|},$$

and so by (iii),

$$\lim_{n \to \infty} |f_n'(b_n)| = \infty. \tag{17}$$

Let now p be a positive integer such that

$$f_{n+2}(c) \geq c, \quad \alpha_{n+1} \geq c, \quad \text{and} \quad |f_n'(b_n)| \geq 2, \quad n = p, p+1, \ldots. \tag{18}$$

Another application of the mean value theorem on $[a_n, b_n]$ yields

$$b_{n+1} - a_{n+1} = |f_n(b_n) - f_n(a_n)| \geq |f_n'(b_n)|\, |b_n - a_n| \geq 2(b_n - a_n),$$

and by iteration,

$$b_n - a_n \geq 2^{n-p} \left(b_p - a_p \right), \quad n = p, p+1, \ldots. \tag{19}$$

From (16) and (19) we get that

$$0 \leq \frac{2^{n-p}(b_p - a_p)}{f_{n+1}^{-1}(c)} < \left| \left(f_n^{-1} \right)' (\alpha_n) \right|, \quad n = p, p+1, \ldots,$$

and so, via (iii)

$$2^{-(p+1)}(b_p - a_p) \lim_{n\to\infty} \frac{2^n}{f_n^{-1}(c)} = 0. \tag{20}$$

(20) and (iv) finally give $a_p = b_p$, which by (13) and (ii) also leads to

$$a_n = b_n, \quad n = p, p+1, \ldots, \quad \text{and} \quad \lim_{n\to\infty} a_n = \infty. \tag{21}$$

Since $\alpha_{p-1} = \left(f_p^{-1} \circ f_{p+1}^{-1} \right)(c)$ and $f_p^{-1}(\alpha_p) = \left(f_p^{-1} \circ f_{p+1}^{-1} \circ f_{p+2}^{-1} \right)(c)$ are both in the domain of $f_1^{-1} \circ \cdots \circ f_{p-1}^{-1}$, if $p > 1$, so is the interval $\left[\alpha_{p-1}, f_p^{-1}(\alpha_p) \right]$. Therefore, $\left(f_p^{-1} \circ \cdots \circ f_{p-1}^{-1} \right)(a_p)$ exists, and then clearly

$$f_{p-1}^{-1}(a_p) = a_{p-1},\ f_{p-2}^{-1}(a_{p-1}) = a_{p-2},\ \ldots,\ f_1^{-1}(a_2) = a_1. \tag{22}$$

Setting now $\mathbf{c} := a_1$, the Foias property follows, as the sequence $x_{n+1} = f_n(x_n), x_1 = \mathbf{c}$, is the only one diverging to ∞. Furthermore,

$$\begin{aligned} \alpha_{n-1} = \left(f_n^{-1} \circ f_{n+1}^{-1} \right)(c) \leq a_n \leq \left(f_n^{-1} \circ f_{n+1}^{-1} \circ f_{n+2}^{-1} \right)(c) = \\ f_n^{-1}(\alpha_n), \quad n = p, p+1, \ldots \end{aligned}$$

implies

$$\begin{aligned} \left(f_1^{-1} \circ \cdots \circ f_{n-1}^{-1} \right) \left(f_n^{-1} \circ f_{n+1}^{-1} \right)(c) \leq \mathbf{c} \leq \\ \left(f_1^{-1} \circ \cdots \circ f_{n-1}^{-1} \right) \left(f_n^{-1} \circ f_{n+1}^{-1} \circ f_{n+2}^{-1} \right)(c), \end{aligned}$$

if $n \geq p$ is odd, since the composition of an even number of f_k^{-1}'s is a strictly increasing function. Thus,

$$\left(f_1^{-1} \circ \cdots \circ f_{2m}^{-1} \right)(c) \leq \mathbf{c} \leq \left(f_1^{-1} \circ \cdots \circ f_{2m+1}^{-1} \right)(c), \quad \text{for} \ \ m \geq \frac{p+1}{2}. \tag{23}$$

Also, the even-term sequence $\left(\left(f_1^{-1}\circ\cdots\circ f_{2m}^{-1}\right)(c)\right)_m$ is increasing while the odd-term sequence $\left(\left(f_1^{-1}\circ\cdots\circ f_{2m+1}^{-1}\right)(c)\right)_m$ is decreasing, for m sufficiently large, as these claims are equivalent to $\alpha_{2m-2}\geq c$, respectively $\alpha_{2m-1}\geq c$. Setting

$$\mathbf{c}^- := \lim_{m\to\infty}\left(f_1^{-1}\circ\cdots\circ f_{2m}^{-1}\right)(c), \quad \text{and} \quad \mathbf{c}^+ := \lim_{m\to\infty}\left(f_1^{-1}\circ\cdots\circ f_{2m+1}^{-1}\right)(c), \tag{24}$$

we have

$$\mathbf{c}^-\leq\mathbf{c}\leq\mathbf{c}^+, \tag{25}$$

and the proof of the Theorem will be complete if we can show that $\mathbf{c}^-=\mathbf{c}^+$.

By way of contradiction, assume $\mathbf{c}^-\neq\mathbf{c}^+$ and let, say, $\mathbf{c}'$ be such that $\mathbf{c}^-\leq\mathbf{c}'<\mathbf{c}$. Then

$$\begin{aligned}&\left(f_{2m+1}\circ\cdots\circ f_1\right)(\mathbf{c}')>\left(f_{2m+1}\circ\cdots\circ f_1\right)(\mathbf{c}) \quad \text{and}\\ &\left(f_{2m}\circ\cdots\circ f_1\right)(\mathbf{c}')\geq\left(f_{2m}\circ\cdots\circ f_1\right)(\mathbf{c}^-)\geq\\ &\left(f_{2m}\circ\cdots\circ f_1\right)\left(f_1^{-1}\circ\cdots\circ f_{2m+2}^{-1}(c)\right)=\alpha_{2m}\end{aligned} \tag{26}$$

(26) shows now that $x_{n+1}=f_n(x_n), x_1=\mathbf{c}'$, also diverges to ∞, a contradiction to the Foias property. $\square$

Remarks. 1) The exact value of the original Foias constant associated to (5) is $\mathbf{c}=\dfrac{1}{\dfrac{1}{\left(\left(\frac{1}{(\cdots)^{1/4}}-1\right)^{1/3}-1\right)^{1/2}}-1}$, while that associated to (6) is $\mathbf{c}=\dfrac{1}{\ln\frac{2}{\ln\frac{3}{\cdots}}}$. Also, thanks to the prowess of Wolfram Alpha [6] a 6-digit approximation of the latter is $\mathbf{c}=1.375892$.

2) It is possible to state a version of the Theorem without reference to the set Σ. Notice that if the hypothesis (i) of the Theorem holds for some number $c>r$, then α_n is well-defined for n sufficiently large. Indeed, α_n makes sense if and only if $f_{n+2}^{-1}(c)>r$, or equivalently $c<f_{n+2}(r)$. Since $f_{n+2}(r)>f_{n+2}(c)$, the claim follows. If now the hypotheses (i) - (iv) are met one can calculate first, as in the Theorem, the Foias number $\mathbf{c}_n$ for the sequence of functions $(f_{n+k-1})_{k=1}^{\infty}$ for some integer n satisfying (vii), and then backtrack from $\mathbf{c}_n$ down to $\mathbf{c}_1=\mathbf{c}$ via the formulas

$$\mathbf{c}_{n-k}=f_{n-k}^{-1}(\mathbf{c}_{n-k+1}), \quad k=1,2,\ldots,n-1. \tag{27}$$

Any $c>r$ could then potentially work.

3) Just as in [2] one can prove that

$$\lim_{n\to\infty}\frac{x_n\ln n}{n}=1, \tag{28}$$

for the sequences $(x_n)_{n=1}^{\infty}$ given by (5) and (6), when x_1 equals the corresponding Foias numbers. $\pi(n)$, the prime counting function [1], also satisfies (28). It is therefore very tempting to inquire whether there are deeper relations between Foias sequences and $\pi(n)$. The answer is a resolute no! If in the Theorem it also happens that $\lim\limits_{n\to\infty} \dfrac{f_n^{-1}(\alpha_n)}{\alpha_{n-1}} = 1$, then $\lim\limits_{n\to\infty} \dfrac{x_n}{\alpha_{n-1}} = 1$, when $x_1 = \mathbf{c}$. A slight modification in example (6), namely $f_n(x) = e^{2n/x}$, gives $\alpha_n = \dfrac{2(n+1)}{\ln 2(n+2)}$, and then for the corresponding Foias sequence $(x_n)_{n=1}^{\infty}$ we have $\lim\limits_{n\to\infty} \dfrac{x_n \ln n}{n} = 2$.

4) One can replace the sequence $(f_n)_{n=1}^{\infty}$ by a new sequence $(f_n^*)_{n=1}^{\infty}$, defined by

$$f_n^*(x) = f_n(x) - r, \tag{29}$$

The new functions continue to be strictly decreasing, with the same domain and range, $(0,\infty)$, so the issue with the domain of a composition of inverses disappears, an advantage. Under mild extra-hypotheses, $(f_n)_n$ and $(f_n^*)_n$ have the Foias property simultaneously, so how do their numbers $\mathbf{c}$ and $\mathbf{c}^*$ relate? The answer is unknown to us! For comparison with 1), we include here the Foias constant associated to the sequence derived from (6), namely $\mathbf{c}^* = \dfrac{1}{\ln\left(1+\frac{2}{\ln\left(1+\frac{3}{\ln(1+\cdots)}\right)}\right)} \approx 1.0097932$.

5) There is an obvious visual analogy between Foias numbers and continued fractions [3]. In fact, if $f_n^{-1}(x) = u_{n-1} + \dfrac{v_n}{x}$, $n = 1,2,\ldots$, $x \in (0,\infty)$, where $(u_n)_{n=1}^{\infty}$ and $(v_n)_{n=1}^{\infty}$ are pre-assigned sequences of positive real numbers, and $v_0 = 0$, then $\left(f_1^{-1} \circ \cdots \circ f_n^{-1}\right)(c)$ approximate the generalized continued fraction $\dfrac{u_1}{v_1 + \frac{u_2}{v_2 + \frac{u_3}{v_3 + \cdots}}}$. A theorem of Sleszynski-Pringsheim [5] guarantees convergence of this continued fraction if $v_n \geq u_n + 1$, $n = 1,2,\ldots$. So, in a certain sense the concept of Foias number generalizes the concept of continued fraction.

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Nicolae ANGHEL,
Department of Mathematics,
University of North Texas,
Denton, TX 76203, USA.
Email: anghel@unt.edu